Σ

Series

Summing it all up

---

28 January 2010

Geometric Series

- Archimedes found that if $-1 < x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

The x = 0 case is undefined in some sense, but the limits work out if we pretend

0^0 = 1

- In Zeno's paradox, $x = 1/2$. There are infinitely many "half distances" to travel, but the total distance and total time are still finite.

Power Series

- Read the other way, we have for $-1 < x < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- That is, we found a series representation for $f(x) = \dfrac{1}{1-x}$.

Power Series

- A power series about x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$$

We can integrate and differentiate power series term by term without changing the radius of convergence.

(Derivative of a sum is the sum of the derivatives; similarly for integrals).

- Note that a power series is a function of x, and so convergence may depend on the value of x. The x values where the series converges form an interval called the interval of convergence. The radius of convergence is half the length of the interval of convergence.

Example

- Find the radius of convergence of the power series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{x^n}{n}$$

This is the power series for the function

ln (1 + x)

- We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\,\frac{n}{x^n}\right| = \lim_{n\to\infty}\left|x\,\frac{n}{n+1}\right| = |x|$$

- Thus the series converges if $-1 < x < 1$ and diverges if $|x| > 1$, so the radius of convergence is 1. (It converges if $x=1$ and diverges if $x=-1$.)

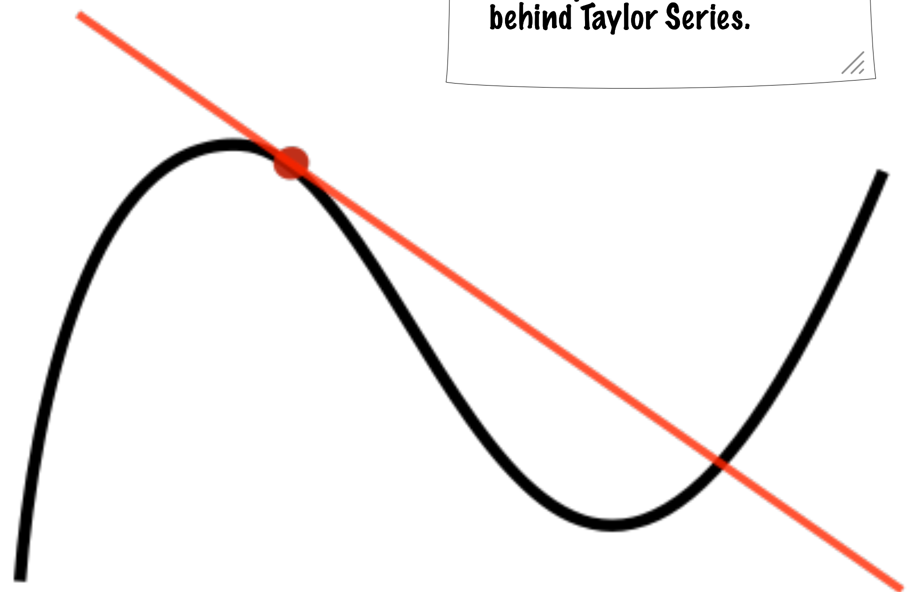
Linear Approximation

- The tangent line locally approximates the function.
- To find $\sqrt{101}$, we let $f(x) = \sqrt{x}$ and take $x_0 = 100$.
- Thus

$$\begin{aligned}\sqrt{101} &= f(101) \\ &\approx f(100) + f'(100)(101 - 100) \\ &\approx 10 + \frac{1}{2\sqrt{100}}(1) = 10.05\end{aligned}$$

If we knew more derivative information, we could use it to approximate the function with a quadratic, cubic, etc... to get better estimates...

this is part of the idea behind Taylor Series.



$$y(x) = f(x_0) + f'(x_0)(x - x_0)$$

http://commons.wikimedia.org/wiki/File:Tangent_to_a_curve.svg

Taylor Series

- If $f(x)$ has a power series expansion about x_0, then it is represented by its Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

There are theorems about when functions have convergent power series expansions, but we won't worry about this for now.

- The Taylor Series with $x_0 = 0$ is also known as the Maclaurin Series.

Example

- Estimate $\sin(1)$ with an error less than .001.

- By the definition of Maclaurin series and some simplification,

$$\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^{2n-1}}{(2n-1)!}$$

The simplification that is involved is that same as in homework 2 when you had to convert the problem with a bunch of zeros into an alternating series.

- Thus,

$$\sin(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

Example

- Estimate $\sin(1)$ with an error less than $.001$.

$$\sin(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

- By the alternating series estimation theorem, if we stop adding at N, we will have an error less than

$$\frac{1}{(2(N+1)-1)!} = \frac{1}{(2N+1)!}$$

Example

- Estimate $\sin(1)$ with an error less than $.001$.

$$\sin(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

- Since $(2N+1)! > 1000$ if $N \geq 3$, we only need to add the first 3 terms:

$$\sin(1) \approx \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} = \frac{101}{120} \approx .84167$$

- For comparison, your calculator will tell you $\sin(1) \approx .84147$.

Estimating Integrals

We can't integrate this problem directly, so we use Taylor series.

- Estimate $\int_0^1 e^{-t^2}\,\mathrm{d}t$ within .001.

- By Maclaurin and some simplification,

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!}$$

- Thus

$$\int e^{-t^2}\,\mathrm{d}t = C + \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(2n+1)}$$

Estimating Integrals

- Estimate $\int_0^1 e^{-t^2}\,\mathrm{d}t$ within .001.

$$\int e^{-t^2}\,\mathrm{d}t = C + \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{n!\,(2n+1)}$$

- By the fundamental theorem of calculus,

$$\int_0^1 e^{-t^2}\,\mathrm{d}t = \left[\sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{n!\,(2n+1)}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$$

Estimating Integrals

If you use fnInt on a TI-83, or the equivalent on your calculator, you will find the integral is about .746824.

- Estimate $\int_0^1 e^{-t^2}\,\mathrm{d}t$ within .001.

$$\int_0^1 e^{-t^2}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$$

- By the alternating series estimation theorem, we will have error less than .001 by adding terms 0 through 4. Thus

$$\int_0^1 e^{-t^2}\,\mathrm{d}t \approx 1 - \frac{1}{1!\cdot 3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7} + \frac{1}{4!\cdot 9} = \frac{5651}{7560} \approx .7475$$

Differential Equations

- A differential equation is an equation that contains an unknown function and one or more of its derivatives.
- In 162.01, we learned how to solve certain types of differential equations, such as those modeling radioactive decay

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -k\,y$$

by the method of separation of variables.

Differential Equations

- Unfortunately, most differential equations cannot be solved by separation of variables.
- One common technique is to solve for the coefficients of a power series solution to the differential equation.
- This strategy requires us to either recognize the power series as a familiar function or be willing to work with its series form.

Example

- Show that

$$y = x + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution to

$$y'' - y + x = 0.$$

Example

- If

$$y = x + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The n=0 term is a constant, so it goes away when we differentiate.

- then

$$y' = 1 + \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- and

$$y'' = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example

- Since

$$y = x + \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- we conclude

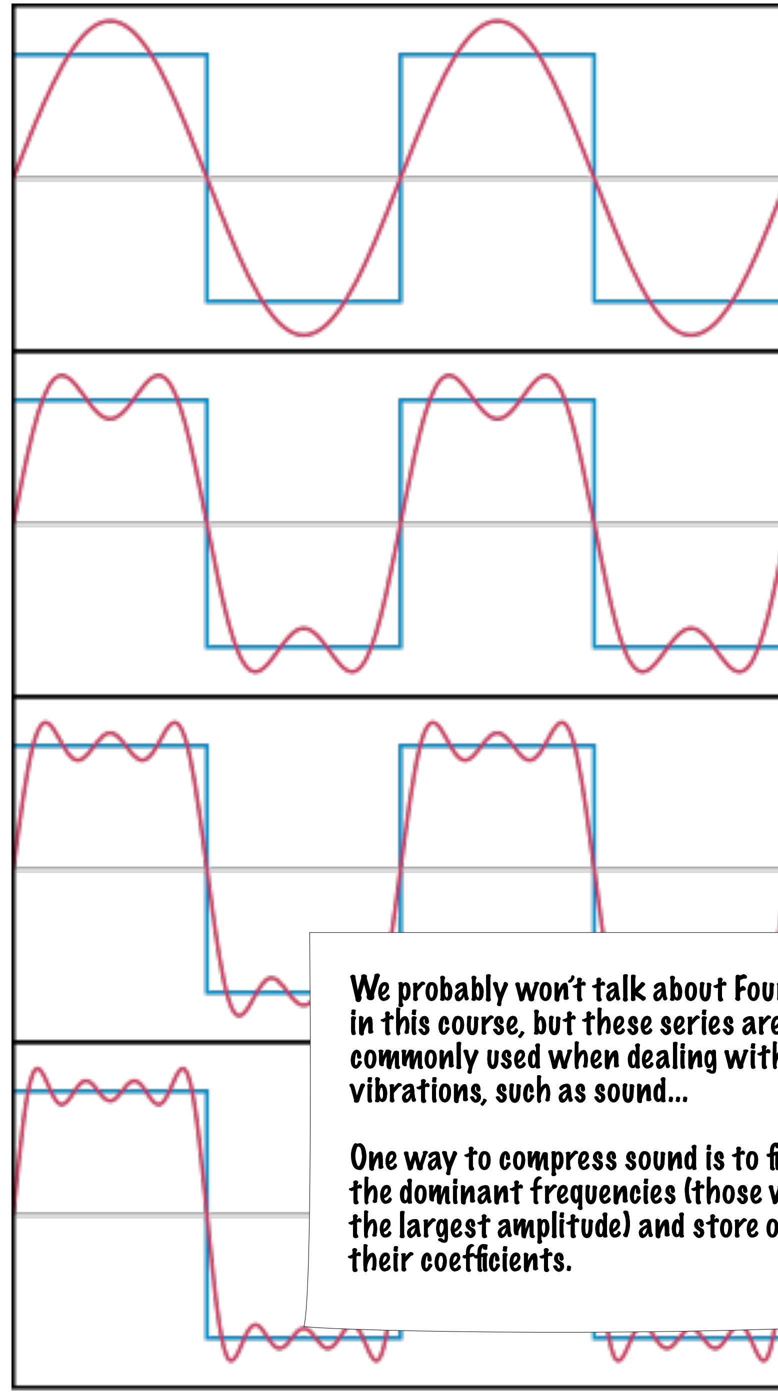
$$y'' - y + x = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) - \left(x + \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) + x = 0.$$

- In this case, we can show $y = x + e^x$.

Fourier Series

- Power series express a function in terms of 1, x, x^2, etc...
- Fourier series express functions in terms of $\cos(1x)$ and $\sin(1x)$, $\cos(2x)$ and $\sin(2x)$, etc...

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,x) + \sum_{n=1}^{\infty} b_n \sin(n\,x)$$



http://commons.wikimedia.org/wiki/File:Fourier_Series.svg

We probably won't talk about Fourier in this course, but these series are commonly used when dealing with vibrations, such as sound...

One way to compress sound is to find the dominant frequencies (those with the largest amplitude) and store only their coefficients.