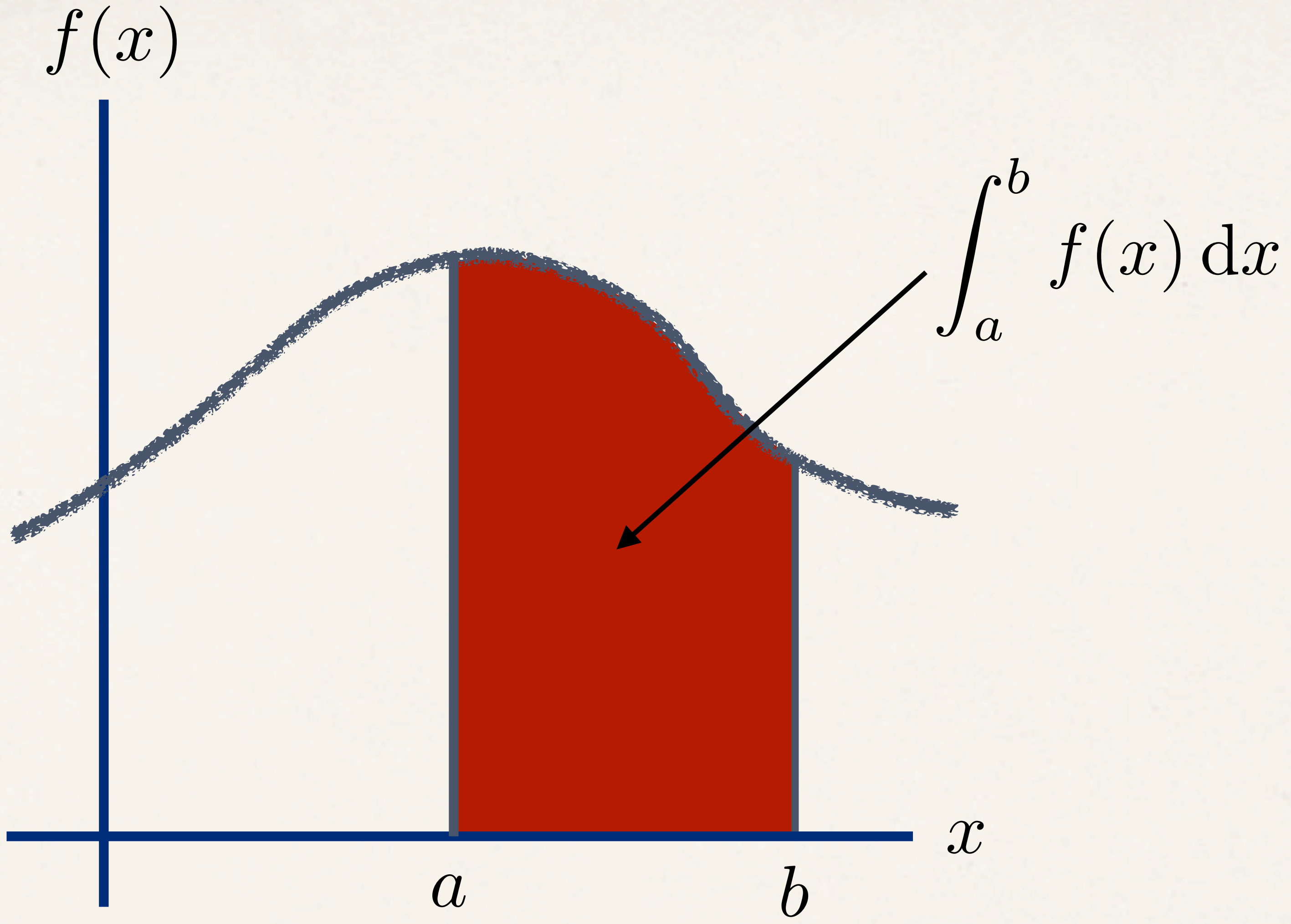

Fundamental Theorem of Calculus

Uniting Integration and Differentiation since the 1600s

8 January 2010

Rates of Change

- Derivatives quantify rates of change.
- Understanding change matters:
  - Populations grow and shrink over time.
  - Ion channels open and close, generating action potentials.
  - Metabolic processes remove medicines from the bloodstream.

Derivatives are great if we know the function and want to know the rate of change.

Sometimes though, rate of change is what we can measure.

The Definite Integral

- If we know the rate of change, we use the definite integral to compute the total change:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\,\Delta x$$

where x_i and delta x are whatever we want them to be, depending on if we use right hand endpoints or left hand endpoints

Example

- A car is initially at rest at a traffic light. Once the light changes, the car accelerates over the next 10 seconds at a rate of $\frac{1}{11}\left(10t - t^2\right)$ m/s^2. How much did its velocity change during those 10 seconds?

The answer is the integral...

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$$\int_0^{10} \frac{1}{11}\left(10t - t^2\right)\,\mathrm{d}t$$

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$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{11}\left(10\left(\frac{(10-0)\,i}{n}\right)-\left(\frac{(10-0)\,i}{n}\right)^2\right)\frac{10-0}{n}\right]$$

Formulas

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} c = n\,c$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad \sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Example

I could have asked about buildup of calcium in a cell; we'd do the same math...

This is also the area under the curve.

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$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{11}\left(10\left(\frac{(10-0)\,i}{n}\right) - \left(\frac{(10-0)\,i}{n}\right)^2\right)\frac{10-0}{n}\right]$$

$$= \cdots = \frac{500}{33}\ \text{m/s}$$

Two problems:

1. This is a lot of work

2. What if I want to know how the velocity changes in 9 seconds? I have to start all over.

A Better Way…

Cumulative Area

- Since the area under $f(x)$ from 0 to b is

$$\int_0^b f(x)\,\mathrm{d}x$$

- we can define a new function which represents the area from 0 to x:

$$F(x) := \int_0^x f(t)\,\mathrm{d}t$$

Cumulative Area

- We can use the cumulative area function to find other areas too, not just the area between 0 and x.

- Suppose, for example, I want to find the area between 1 and 2. We know the area from 0 to 2 is the sum of the area from 0 to 1 and the area from 1 to 2:

$$\int_0^2 f(t)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t + \int_1^2 f(t)\,\mathrm{d}t$$

- Thus:

$$\int_1^2 f(t)\,\mathrm{d}t = \int_0^2 f(t)\,\mathrm{d}t - \int_0^1 f(t)\,\mathrm{d}t = F(2) - F(1)$$

Cumulative Area

- In general, if F is the cumulative area function for f, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

If we know F, this lets us solve the generalization problem... but how can we find F? This seems like a harder problem than we had before...

Fundamental Theorem, Part I

- Suppose f is continuous on $[a, b]$.
- Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then $F'(x) = f(x)$.
- That is, the rate of change of the area under the curve f is equal to f.

This also means that the cumulative area function is _an_ antiderivative.

But remember from 151 that all antiderivatives only differ by a constant, so when we subtract to find the area between a and b, this goes away. Thus, any antiderivative will do.

Fundamental Theorem, Part II

- Suppose f is continuous on $[a, b]$.
- Let F be any antiderivative of f.
- Then $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

Example

- Find:

$$\int (3x^2 - 2x)\ \mathrm{d}x$$

add an example involving taking the derivative of a cumulative area function.

Example

- Let $F(x) = \int_2^x \cos\left(t^2 - 1\right)\, \mathrm{d}t.$

- Find $F'(x)$.

This is a trivial application of the fundamental theorem of calculus part I.

Example

- Find:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{x^3+1} t\,\sin(t)\,\mathrm{d}t$$

This more complicated example requires the use of the chain rule.